Introducing Four Models for the Set of All Sounds and Constructing Common Musical Notation Using Them

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Abstract

The typical approaches to formulating music theory mathematically neglect the fundamental properties that allow us to use mathematical tools. We introduce these properties in the form of sets, building four models for a so-called *set of all sounds*. Using this, we give a mathematical framework that can be used to recreate music theoretical ideas. This further allows for the creation of new musical ideas. After defining the set of all sounds, we can then manipulate it to exhibit its basic mathematical properties and demonstrate how they can be used to build up sheet music and other musical constructions.

1 Introduction

It is said that Pythagoras came back from Babylon and introduced harmony to Greece; and so music was born. In this paper, we explore the mathematical properties of sound and music for both the mathematician and the musician. It perhaps is not mathematically rigorous enough for the mathematicians and perhaps it lacks the musical rigor found in typical music theory papers. But it does possess an ability to express a combined viewpoint that is becoming more popular among those trying to understand this multidisciplinary topic. Some may refer to it as Mathematical Music Theory, Mathemusic, Musical Set Theory, Axiomatic Music Theory or simply Math and Music. We will refer to it as Musical Mathematics.

Perhaps the mathematician views his work as music- a beautiful piece of art that makes one want to dance and sing aloud. And perhaps the musician views his work as mathematical- a beautiful piece of exactness and structure that allows for much deeper exploration into his work. But the humble objective for this paper is to show that while music and math are both inherently beautiful, the combination is within a dimension that redefines beauty and promotes the beauty of each individually. Further, the exploration of this is done from a perspective that is inherently mathematical, yet has the ability to appreciate the significance of musical ideas. While this paper is simple enough so that anybody with a curious mind can understand its ideas, it is written from the point of view of mathematics.

There is perhaps a need for a basis of reasoning for the purpose, motivation and significance of a presentation of seemingly unrelated fields. The interdisciplinary excursion on its own might be enough to warrant an exploration into this topic, but a more reasonable approach is the following proposal.

Exploration into this field is an exercise in the ability that mathematics has in explaining nature. It further cements mathematics' role into everyday life and provides further evidence for its significance. This exercise is about using well-established mathematical tools to explain something that is already known to have mathematical properties, but is perhaps not fully developed. It is our belief that these tools are able to provide sufficient use in establishing a foundation for a field that is becoming increasingly mathematical.

This reasoning on its own provides a tangible motivation to spark one's scientific curiosity. However, to give the reader more insight into the motivations of our work, we will try to serve it with our anecdotal observations. Music has always been known to possess properties of mathematics, whether we understand what those are or not. Further inquiry into the methods of music, i.e., learning to play an instrument or learning how to read sheet music, will, to the curious mind, create interest into the structure, the construction and the reasoning to how music is built up into beautiful works of art. Questions arise- to the physicist, perhaps they wonder how the sound is produced from different instruments or how the wave mechanics interact to produce sound; to the mathematician, perhaps it is reasonable to assume that they would inquire about the structure of sound elements and what properties it possesses; to the musician, they might wonder how harmonies are formed and how the fundamentals of music theory are constructed. These, we think, provide motivation for a serious look into these ideas. Perhaps this may be insufficient for some, but it has provided us with the right sparks to pursue them.

The significance of such ideas express the ability that mathematics has to connect seemingly unrelated fields. Connecting fields that are disparate, and creating spheres of interdisciplinary exploration, opens up other areas of work that give us a better understanding of the world around us. It might provide the mathematician with more fundamental understanding of musical ideas and perhaps provide the musician with a more mathematically-minded approach to their work. Of course, one may argue that the intermingling between the arts and sciences leaves both worse off at the end of such endeavors. But those with reasonable mindsets will realize that science is partly an art and that art is partly a science. The connection between the two is hardly undeniable and perhaps the distinctions come only from the universities who separate them by college and degree.

The connections that music has to mathematics has been realized by both musicians and mathematicians alike. Further, the physical significance has also been rigorously explored by physicists and other scientists. There have been numerous books on the subject of Musical Mathematics, and there have been some interesting recent work discovering the fundamentals of these connections. In our own treatise, we will explore the basic ideas of Musical Set Theory. In this approach, we explore four models that build on the ideas of how the set of sounds is constructed.

2 Modelling Sound

Just like complex mathematics, music is built from basic building blocks. In this case, music can be overtly boiled down to be a set of sounds played in sequence. Therefore, we must define this set of sounds so that we can use it to define what we know about music. One might consider music to be sound that is of a particular flavor- notes of a set frequency that yield endless amounts of expression. Yet this perspective underscores the value of a general set of sounds. We must consider a version of sound that can explain most, if not all sound phenomena. In step with asking about the basic properties of numbers, we can also inquire into the basic properties of sound. Sound possesses four fundamental properties- frequency, amplitude, spectrum and time. To the musician, these are pitch, loudness, timbre and length, respectively. Since we take the mathematician's point of view, we will consider the scientific denominations for these properties.

We then can consider what we can do with these properties when constructing our set of sounds. Our goal is to create a set of all sounds that not only embodies all of music, but also contains every possible sound, both audible and inaudible. We can do this by considering four models of sound that work the properties of sound into coherent ideas and formulations. There are many models that can be created using these four properties so there is need to eliminate some trivial and useless cases. We realize that using only one of the properties described above is particularly inefficient when describing the set of all sounds since we cannot make sense of one property without at least one more. For instance, we cannot consider frequency alone as it may be able to describe the pitch of a sound but not necessarily how that sound is interpreted physically. The physics of sound is ultimately what we must rely on to construct our models sensibly. Of course, one can endlessly consider models of a single variable, but one cannot construct anything meaningful without context. In our case, the context is provided by considering other properties of sound. When considering a set of all sounds, we must acknowledge the features that are inherent about it.

We attempt to do this in the four models we have built. Each model takes different properties of sound into account which allows us to conclude which set is most sensible to be used.

2.1 Model 1

The first model that we consider, has two properties of sound, that of frequency and amplitude. We will say that the set of sounds that we are trying to find is inherently two-dimensional. Furthermore, we can claim that it must be a two-dimensional set of real numbers so that we have a pair of numbers that represent frequency and amplitude. Mathematically, this has the form

$$\mathbb{S}_1 = \mathbb{R}^2 = \{(f_i, dB_i) : f_i, dB_i \in \mathbb{R}\},$$

where $\mathbb{S}_1$ represents the first model for the set of all sounds, f_i is the frequency, dB_i is the amplitude, and $\mathbb{R}$ is the set of real numbers. We also note that $\mathbb{R}^2$ is the two-dimensional set of real numbers.

One might inquire into the reason that we label coordinates of $\mathbb{R}^2$ as f_i and dB_i where any other variable could be sufficient. We note that we are considering physical phenomena and so our labelling is purposeful. The labelling that we use is to make the connections between the real numbers and physical quantities more apparent and gives the reader an understanding of why and how these concepts that we will be presenting come about.

We will note that the amplitude and frequency of a sound is determined by external considerations such as the physics of musical instruments and sound. When considering these physical phenomena, we note that the amplitude cannot be less than zero. We say that it is not physical for the amplitude to possess this property. However, one might consider imaginary amplitudes or negative amplitudes in another treatise of Musical Set Theory.

We will also say that the frequency of a sound cannot be less than zero. However, a consideration of negative or imaginary frequency will perhaps be useful in another approach to this theory such as considerations in signal processing.

In this treatise of sound, we will also consider the fact that when frequency or amplitude is zero, a sound makes no sound. That is, we do not consider it a sound when frequency or amplitude is less than or equal to zero. This implies that the set of all sounds is made up of sounds that are in our physical space. This space is inherently physical, i.e, it possesses properties that exist naturally and exists even without the presence of ears to hear it. We consider this space to be similar to space-time. For time still exists even if it has not been experienced by the human or animal consciousness. So too do we consider seemingly inaudible sounds with frequencies and amplitudes that tend towards infinity.

Therefore, we can redefine our set of all sounds to be the set of ordered pairs of positive real numbers, i.e.,

$$\mathbb{S}_1 = \mathbb{R}^{+2} = \{(f_i, dB_i) : f_i, dB_i \in \mathbb{R}^+\},$$

where $\mathbb{R}^+$ is the set of positive real numbers and $\mathbb{R}^{+2}$ is the two-dimensional set of positive real numbers.

This definition of the set of all sounds provides us with our first model for the set of all sounds. Now the question becomes, how can we compare elements in $\mathbb{S}_1$? We will first have to classify two terms of comparison when working with sound. The first is comparing pitch. We will say that a sound is *lower-pitched* if the frequency of one sound is less than that of another sound. Likewise, we say that a sound is *louder* if the amplitude of one sound is greater than that of another sound. Consider two sounds, s_1 and s_2 which are elements of $\mathbb{S}_1$, i.e., they have the form $s_1 = (f_1, dB_1)$ and $s_2 = (f_2, dB_2)$. Then s_1 is lower-pitched than s_2 if f_1 is less than f_2. Similarly, s_1 is louder than s_2 if dB_1 is greater than dB_2.

We note that this model of the set of all sounds considers two physical phenomena that are connected to musical phenomena. That is, $\mathbb{S}_1$ possesses properties of "pitch" (frequency) and "loudness" (amplitude). Because of this, we can consider more relations on our model that fully encapsulates the comparison of two distinct sounds.

This results in the following relations that we can create when comparing two sounds. Again consider s_1 and s_2 to be elements of $\mathbb{S}_1$ that are defined as $s_1 = (f_1, dB_1)$ and $s_2 = (f_2, dB_2)$. Then we define the relation "$=_1$" on $\mathbb{S}_1$ as follows: $s_1 =_1 s_2$ if $f_1 = f_2$ and $dB_1 \neq dB_2$. Similarly, we define "$=_2$" on $\mathbb{S}_1$ as follows: $s_1 =_2 s_2$ if $f_1 \neq f_2$ and $dB_1 = dB_2$. We can also define the relation "$<_1$" on $\mathbb{S}_1$ as follows: $s_1 <_1 s_2$ if $f_1 < f_2$ and $dB_1 \geq dB_2$. Likewise, we define the relation "$<_2$" on $\mathbb{S}_1$ as follows: $s_1 <_2 s_2$ if $f_1 \geq f_2$ and $dB_1 < dB_2$.

We note that the relations "$=$", "$<$" and "$\geq$" used above are the usual relations for the set of two-dimensional real numbers. With these definitions, we are able to compare elements of $\mathbb{S}_1$ as we do so musically. We now consider extensions of this model in the forthcoming frameworks.

2.2 Models 2 and 3

In these models, we define a third coordinate of Model 1 as part of our construction for the set of all sounds. The first that we will consider is a set of all subsets of a given set. In this case, the set is the set of harmonics. This can be interpreted as the spectrum of sound, or timbre. That is, for each sound, we define a set of harmonics for that sound. Then the set of all sounds can be defined as the set of ordered triplets so that we have a representation of frequency, amplitude and set of harmonics, i.e.,

$$\mathbb{S}_2 = \{(f, dB, A_f) : f, dB \in \mathbb{R}^+, A_f = \{kf : k \in \mathbb{Z}^+\}\} \subseteq \mathbb{R}^{+2} \times \mathcal{P}(\mathbb{R}^+),$$

where $\mathcal{P}(\mathbb{R}^+)$ is the set of all subsets of the set of positive real numbers and A_f is the set of harmonics for the given frequency, where k is a positive integer and $\mathbb{Z}^+$ represents the set of all positive integers. We note that $\mathbb{S}_2$ represents the second model of the set of all sounds.

For Model 3, we will use the property of time as the third coordinate for the set of all sounds, i.e., length in music. Then the set of all sounds is given as the set of ordered triplets so that we have a representation of frequency, amplitude and time. Mathematically, this is

$$\mathbb{S}_3 = \mathbb{R}^{+2} \times \mathbb{R}_{\geq 0} = \{(f, dB, t) : f, dB \in \mathbb{R}^+, t \in \mathbb{R}_{\geq 0}\},$$

where $\mathbb{S}_3$ represents the third model of the set of all sounds, $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers. We have this as a possible value for t since we consider time to physically start from zero.

However, there is a nuance with this model. To build anything meaningful from this set, we must consider a function, μ that maps a time interval $[a, b]$ to $\mathbb{S}_1$. That is, $\mu : [a, b] \to \mathbb{S}_1$, where $\mu(t) = (f(t), dB(t))$. This will be crucial in deciphering sounds that occur one after the other.

We can now say a few things about this model of the set of all sounds. For a sound, $s(t) = (f(t), dB(t), t)$ in $\mathbb{S}_3$, we say that $s(t)$ is a *constant sound* if for all t in the interval from a to b, the frequency and amplitude are constant. This can be condensed as a set notated by C_s, where for each point in time, the frequency and amplitude remain the same. That is, $f(t) = f$ and $dB(t) = dB$ for all $t \in [a, b]$.

Then we say that C_s is *transient* if there is some value for t in which the frequency or amplitude changes. Similarly, we can say that C_s is *properly constant* if C_s is a set with infinitely many values. Of course, there is a nuance in the case that amplitude is dependent on time. Here we can say that as time goes to infinity, the amplitude goes to zero. Note that we are assuming that the strength of the amplitude decreases as time goes on which is intuitive enough to realize from physical phenomena.

Now suppose that we have two sounds in the set of all sounds, s_1 and s_2 so that $s_1 = (f_1, dB_1, t_1)$ and $s_2 = (f_2, dB_2, t_2)$. Then we say that s_1 and s_2 are *simultaneous sounds* if $t_1 = t_2$. That is, s_1 and s_2 are "played" at the same time. Now if we consider time to always be a constant value, we will realize that we are left with just variable frequency and amplitude, i.e., a two-dimensional subset of $\mathbb{S}_3$. Therefore, when time is constant, we obtain

$\mathbb{S}_1$, i.e., Model 1. This means that $\mathbb{S}_3$ is a generalization of $\mathbb{S}_1$.

We can also consider these two-dimensional subsets when we freeze time, that is, consider a chosen time. Then we call the set of all sounds for this time to be a *sound frame*.

2.3 Model 4

In this final model that we will look at, we consider the combination of previous models. Namely, we take all four attributes of what makes a sound- frequency, amplitude, spectrum and time, and use them to construct our set of all sounds. Then the set of all sounds is given as the set of ordered quadruples so that we have the representation of frequency, amplitude, spectrum (harmonics) and time in each coordinate, i.e.,

$$\mathbb{S}_4 \subset \mathbb{R}^{+2} \times \mathbb{R}_{\geq 0} \times \mathcal{P}(\mathbb{R}^+)$$

$$\text{and } \mathbb{S}_4 = \{(f_i, dB_i, t_i, A_{f_i}) : f_i, dB_i \in \mathbb{R}^+, t_i \in \mathbb{R}_{\geq 0}, A_{f_i} \in \mathcal{P}(\mathbb{R}^+)\},$$

where $\mathbb{S}_4$ represents the fourth model of the set of all sounds and frequency, amplitude, set of harmonics and time are the same as in previous models.

2.4 Comparing Models

In the above sections, we have defined four models for the set of all sounds. We now have the ability to compare these models to see which is the best to use to build further constructions and fit reality most accurately.

The first model provides us with two attributes of sound- frequency and amplitude. These correspond to pitch and loudness, respectively. This is a good way to describe sound two-dimensionally, and provides us with the ability to form sounds with harmonics. However, this model does not allow for new sounds to be uniquely heard. That is, we can only append the array of sounds that exist in our set of sounds. We cannot play music, and further, we cannot hear distinction. With this in mind, the low-dimensional space that $\mathbb{S}_1$ exists in is useful to define functions and build algebraic structures in.

In the second model, we define three attributes, frequency, amplitude and spectrum. That is, we are implementing the set of harmonics for a given frequency f. This is helpful since we do not need to use a function to define the harmonics. Rather, we use the set of all subsets of the positive real numbers, and use them as an attribute for the set of all sounds. However, like Model 1, it does not provide a consecutive motion of sound, and distinction is not necessarily clear.

In Model 3, we introduce time as the third attribute of the set of sounds. This allows us to have sounds that are heard consecutively and uniquely. It also allows us the opportunity to look at singular time frames which mimic Model 1 and define algebraic structure for the set of harmonics. It is this set that we will be considering to be $\mathbb{S}$, our general set of all sounds. This set provides the most flexibility and ease of use for defining algebraic structure.

The final model provides us with all the attributes of what makes a sound. That is, it gives us the pitch, loudness, duration and timbre. Timbre is an important feature of sound as it provides the distinction we need between notes from different instruments and defines a set of harmonics. However, we believe that although the timbre is a fundamental part of musical sound, we can reconstruct it using the three other properties of sound given by time, frequency and amplitude. We also note that in order to define distance on the set of all sounds, we cannot have the set of all subsets along with sets of size much much less than the set of all subsets of those sets. We note this importance since we will use distances to define and build musical constructions.

3 Simple Mathematical Constructions

Now that we have these models defined, we can consider simple mathematical constructions. These constructions will be considered only for Model 1, i.e, $\mathbb{S}_1$ since it is the simplest to build from basic ideas in mathematics. By just considering frequency and amplitude, we can form the fundamentals to then consider for other models.

Often in music, the frequency or pitch of a sound is varied without any concern for the amplitude. These are considered in harmonics, octaves and other musical constructions. We begin with simple mappings, that is, functions to vary the frequency by a fixed real number.

3.1 Mappings

The first function that we consider takes a sound from $\mathbb{S}_1$ and multiplies the frequency by a fixed real number given by c which produces another sound. That is, we map a sound into another sound. In other words, if s_1 is a sound in $\mathbb{S}_1$ so that $s_1 = (f_1, dB_1)$ and a positive real number c, we define $g_c : \mathbb{S}_1 \to \mathbb{S}_1$ as follows:

$$g_c(s_1) = (cf_1, dB_1).$$

We say that g_c form a family of functions in which we call *frequency variant*.

Now within our set of all sounds, we realize that there is a subset that contains sounds that are audible to the human ear. We will consider this subset to be $\mathbb{S}_H$. Clearly, the size of $\mathbb{S}_H$ is smaller than $\mathbb{S}_1$ and we call it the set of all *audible* sounds when the frequency and amplitude are within the range of common human hearing. Further, to generalize this realization, we can say that there are other subsets of $\mathbb{S}_1$ that have any form containing inaudible or audible sounds or both. We call these subsets *soundscapes*. These soundscapes may be useful for signal processing and other constructions of sound. However, for simplicity, we will only consider the soundscape of audible sounds.

This prompts the idea to start considering music within $\mathbb{S}_H$. The most basic idea in music is that of a note. We can consider what this means generally, but to keep the trend of considering only audible sounds, we consider notes that are commonly known in music. For a commonly known frequency, f, a *note* of this frequency is a sound $s_i = (f_i, dB_i)$ so that $f_i = f$. That is, we have a sound where the amplitude can change, but the frequency remains the same. That is, the pitch is fixed with the loudness able to fluctuate. We note that this means two sounds with the same frequency but differing amplitudes results in two different notes. The set of all notes of frequency f is given by N_f which is equal to the set $\{(f, dB_i) : dB_i \in \mathbb{R}^+\}$, where dB_i is a positive real number. We now realize that N_f is what we typically know as a musical note. That is, a musical note is of one frequency but can have an amplitude that is any positive real number.

Then we obtain a set of all commonly known notes in music that are formed using N_f. We can now define functions using this notation for notes that are functions of the family of functions we defined earlier as frequency variant. The first function to consider is that of harmonics. We will let g_h denote our function that is a mapping from all audible sounds to all audible sounds. That is, $g_h : \mathbb{S} \to \mathbb{S}$ where the frequency is multiplied by a positive integer. We realize that this will form a set of harmonics for a given frequency. Mathematically, this is interpreted as

$$g_h(N_f) = (h \cdot f, dB_i),$$

where h is a positive integer and dB_i is the amplitude contained within the set of positive real numbers. We call this family of functions, the family of *harmonic functions*. The range of g_h is realized to be the set of harmonics for a given note. We now build other simple mathematical constructions using $\mathbb{S}_1$.

3.2 Operations

When considering operations, the first consideration is an "addition of sounds". What this means is simply having two notes being "played" at the same time. In this model, everything is in the same time frame, since we do not consider it, but we would like to combine sounds rather than having one element being played. To define an addition, we need to consider the union of two sounds. This is simply two sounds in one set together. This representation is the formal expression of having one of multiple sounds. Under normal addition in $\mathbb{R}^2$, we obtain the addition of coordinates. This does not make sense in our context of combining sounds. For two frequencies played individually does not imply that the addition of those frequencies is perceived to be the same or even is the same. Therefore, we must define the union of two sounds to be the first operation on $\mathbb{S}_1$. However, in order to have this operation, we must consider the set of all subsets of $\mathbb{S}_1$ and subsequent subsets of this set to be individual sounds.

That is, we consider the set of all finite subsets of $\mathbb{S}$. Meaning that this subset does not have infinitely many sounds contained within it. This can be represented mathematically as $\mathcal{S} = \{A \subset \mathbb{S} : A \text{ is finite}\}$. Then we can consider the operation, $\cup$ on $\mathcal{S}$ with two subsets of $\mathcal{S}$, s_1' and s_2' that combines the elements from each subset. This will give us the set that contains s_1' and s_2'. That is, $s_1' \cup s_2' = \{s_1', s_2'\}$.

We can now redefine μ as a function of time that maps a point in a time interval to the set $\mathcal{S}$. That is, $\mu : [a, b] \to \mathcal{S}$ so that $\mu(t) = \{s_1'(t), ..., s_n'(t)\}$. We note that $\mu(t) = A(t)$, where A is a finite subset of $\mathcal{S}$.

3.3 Semigroups

The last simple mathematical construction that we will consider is that of semigroups. A semigroup is just a set with an operation that is associative. Associativity just means that rearranging parenthesis in an expression will not change the end result. For example, in the set of real numbers, addition is an associative operation. From basic theory, we know that the operation, $\cup$, is an associative operation. Therefore, $\mathcal{S}$ along with $\cup$ is a semigroup.

However, we can extend this idea further by considering other properties of the operation,

$\cup$. We know that $\cup$ is commutative (changing the order of the operands does not change the result) and the identity element (leaves an element unchanged when combined with the identity) is the empty set. The empty set is a set that contains nothing and notated as ϕ. So the union of the empty set with any element of $\mathcal{S}$ produces that same element. That is, s' in $\mathcal{S}$ returns s' when $s' \cup \phi$.

Given the fact that $\cup$ is associative, commutative and there exists an identity element for the semigroup, we obtain the fact that $\mathcal{S}$ along with $\cup$ forms a semilattice with identity. We can now describe a semigroup homomorphism using the previously defined function, g_c that multiplies the first coordinate of a sound by a real number c. A homomorphism is simply a mapping between two sets that preserves the preserves the operations on that set. In the context of our semilattice, we redefine g_c as ϕ_c as a function from $\mathcal{S}$ to $\mathcal{S}$ so that for s' in $\mathcal{S}$ we have

$$\phi_c(s') = \{(c \cdot f, dB) : (f, dB) \in s'\}.$$

One can easily confirm that this is a homomorphism on $\mathcal{S}$.

4 Constructing Music

The most basic musical construction that we can attempt to build is one that is well-known and used often, the musical staff, i.e., sheet music. Any musician or mathematician with a keen eye will notice that there is much structure built into the staff and for good reason. The staff provides a structure that is consistent and well-defined. The question arises, why try to build a staff mathematically when it already exists and its structure is well-known? The answer to this question lies in the fact that we will be able to build this using our defined set of all sounds, providing evidence that it is a useful and workable set. It is also good practice to be able to build something that is already known based on the mathematical fundamentals behind it. Further, since we are reverse-engineering the staff, we can then observe what other constructions we can make from our set of sounds that are more generalized.

4.1 Metric

Since we define frequency and amplitude as elements of the positive real numbers, we can define a Euclidean metric, i.e., the typical distance function that exists for real numbers. However, when considering the third coordinate of the set of all sounds, time, we must recognize three features of sheet music. These are note length, time signature and tempo. To measure time, we must take these three into consideration. In order to model sheet music, we will notice that each note is characterized as a discrete event in continuous time. Therefore, we must model our function after this. Because of this, there is also is a possible need for introduction of discrete-event continuous stochastic processes, but this is saved for an exploration later on. A good treatise of this has been done by Iannis Xenakis in his book, *Formalized Music*.

Because of these considerations with discrete events, we can have two distance functions. The first simply treats time as a subset of real numbers greater than or equal to zero. The second is using our μ function that we defined earlier that maps a time interval onto $\mathbb{S}_1$.

For our other variables, amplitude and frequency, we can simply use a Euclidean metric as mentioned earlier. The distance between two amplitudes, dB_1 and dB_2, is just a positive real number and the distance between two frequencies is also a positive real. We can now generalize our metric that describes the distance between two sounds. For sounds s_1, s_2 in $\mathbb{S}$, where $s_1 = (f_1, dB_1, t_1)$ and $s_2 = (f_2, dB_2, t_2)$ we have

$$d(s_1, s_2) = c_1|f_1 - f_2| + c_2|dB_1 - dB_2| + c_3|t_1 - t_2|,$$

where c_1, c_2, c_3 are real constants. Then we can also define a metric with μ. That is, for functions μ_1 and μ_2, we have

$$d(\mu_1, \mu_2) = \int_a^b d(\mu_1(t), \mu_2(t))\, dt$$

and

$$d(s_1, s_2) = c_1|f_1 - f_2| + c_2|dB_1 - dB_2|,$$

where c_1, c_2 are real constants. Then note that we have two distance functions when we consider μ that describe distance in $\mathbb{S}$.

4.2 Sheet Music

Given this defined metric, we can now define what we know as common musical notation. We say that if $|f_1 - f_2|$ is equal to the distance between two consecutive commonly known notes, then the metric is a *normal notation metric*. However, there is now the need for diatonic theory. The common distance between consecutive notes in an octave scale is $2^{1/12}$. That is, if f_1 and f_2 are frequencies for consecutive common notes, then the distance between them is given by $d(f_1, f_2) = |f_1 - f_2| = 2^{1/12}$. Using this distance, we can construct sheet music.

In music, the distance between notes is usually represented as lines and spaces between the lines. In each line or space, a note is placed that describes the pitch. So too, the distance function that we consider can create a line in space for every note that is $2^{1/6}$ apart. Musically, this is considered a whole-step while distances of $2^{1/12}$ are known as half-steps. Then using the distance function for time, we can define commonly known time signatures, note lengths and tempo.

Along with variables of time, we can manipulate the amplitude and frequency to obtain different characteristics of sheet music. For instance, we can create a function that increases the amplitude gradually as time goes on and notes are played. Musically, this is of course known as a crescendo. These and other extensions will be discussed in the next section.

5 Discussion and Conclusions

We have introduced four models to build a framework for the set of all sounds. We showed that it resembles sets that are commonly used in mathematics and shares properties of sound. We also built simple constructions to exhibit the simplicity in which our models are built. From there, we built a distance function that allows us to build sheet music. In Model 1, we considered frequency and amplitude which we then used to create simple mathematical constructions. In Model 2, we established the set of harmonics for each frequency in the set of all sounds. The model that we accepted as our resolute was Model 3 which contained in addition to frequency and amplitude, time. This model, we argue provides the most logical and realistic framework for building the set of all sounds. We realize that it contains the ability to represent the set of harmonics using functions rather than a fourth coordinate. Finally, the fourth model presented all attributes of sound into one set. Like our argument for why Model 2 is not the best candidate, Model 4 shares the same redundancy problems that are present in Model 2.

The final set of all sounds is therefore used in the creation of our distance function. We note that there are many extensions that can be done mathematically to recreate common sheet music notations. This can be done by considering what the notation in the score does to a particular note. We can question if it changes the pitch, loudness or length of the note. This then allow us to interpret the notation into terms that we can understand mathematically and therefore result in typical musical notation.

With this in mind, we can also consider other generalizations of sheet music. For instance, suppose that we set the distance function so that we consider other constant variables for frequency. That is, instead of using $2^{1/12}$, we can consider any other positive real number. This would then result in a new form of sheet music that has its own properties of sound and music. There is a nuance in the fact that we do not just consider audible sound when we talk about sheet music, we also rely that what we hear as a result to be music. This means that our generalization for all forms of sheet music may only have a few cases in which the music is pleasant and interesting. However, we nonetheless can obtain a generalization for this for other soundscapes with other metrics.

These can possibly be explored using digital signal processing which gives the user the ability to hear what another construction of sheet music would sound like. Furthermore, this method may not be as fruitful when considering soundscapes with inaudible sounds. However, mathematical formulations can still be obtained and so digital signal processing would still merit results in this area of exploration.

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