Introducing Four Models for the Set of All Sounds and Constructing Common Musical Notation Using Them

Exploring Mathematics and Music

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Motivations

- Lack of Academic Work in Musical Set Theory

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- An Exercise in Mathematics

Models of Sounds

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Model 1

Define $\mathbb{S}_1$ to be the set of ordered pairs of positive real numbers with frequency f and amplitude dB. That is,

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Model 2

Define $\mathbb{S}_2$ to be the set of ordered triplets with representations for frequency, amplitude and set of harmonics. That is,

$$\mathbb{S}_2 = \{(f, dB, A_f) : f, dB \in \mathbb{R}, A_f = \{kf : k \in \mathbb{Z}^+\}\} \subseteq \mathbb{R}^{+2} \times \mathcal{P}(\mathbb{R}^+).$$

Models of Sounds (continued)

Model 3

Define $\mathbb{S}_3$ to be the set of ordered triplets with representations for frequency, amplitude and time. That is,

$$\mathbb{S}_3 = \{(f, dB, t) : f, dB \in \mathbb{R}^+, t \in \mathbb{R}_{\geq 0}\} = \mathbb{R}^{+2} \times \mathbb{R}_{\geq 0}.$$

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Model 4

Define $\mathbb{S}_4$ to be the set of ordered quadruples with representations for frequency, amplitude, time and set of harmonics (spectrum). That is,

$$\mathbb{S}_4 = \{(f, dB, t, A_f) : f, dB \in \mathbb{R}, t \in \mathbb{R}_{\geq 0}, A_f \in \mathcal{P}(\mathbb{R}^+)\}.$$

Some Simple Properties

Frequency Variant Family of Functions

We construct the following mapping for Model 1 on $\mathbb{S}_1$: Let $s \in \mathbb{S}_1$ such that $s = (f, dB)$. Then we define $g_c : \mathbb{S}_1 \to \mathbb{S}_1$ as follows:

$$g_c(s) = (cf, dB),$$

where c is a positive real number.

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Define the power set of $\mathbb{S}_1$ to be $\mathcal{S}$ with the condition that the subset is finite. That is, $\mathcal{S} = \{A \subset \mathbb{S}_1 : |A| < \infty\}$.

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Combination of Sounds in Model 1

Take two subsets of $\mathcal{S}$, say S_1, S_2 such that $S_1 = \{s_{i_1}, ..., s_{i_k}\}$ and $S_2 = \{s_{j_1}, ..., s_{j_n}\}$. Then define the sound $S_1 \cup S_2$ as follows:

$$S_1 \cup S_2 = \{S_1, S_2\}.$$

Note there is an abuse of notation.

Metric on Model 3

We define the metric on $\mathbb{S}_3$ for sounds s_i, s_j so that $s_i = (f_i, dB_i, t_i)$ and $s_j = (f_j, dB_j, t_j)$ as follows:

$$d(s_i, s_j) = c_1|f_i - f_j| + c_2|dB_i - dB_j| + c_3|t_i - t_j|,$$

where c_1, c_2, c_3 are positive real numbers.

Musical Constructions

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If $|f_i - f_j| = 2^{1/12}$, then we obtain the typical structure for sheet music.

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Generalized Sheet Music

Define a fixed frequency metric, so that $|f_i - f_j| = c$ for some fixed value c.

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